Sample Midterm Questions (CS 477 Spring 2013)

1. Give truth tables for each subformula of the following formulae:

   (a) $\varphi_1 : A \Rightarrow A \wedge (A \vee B)$

A	B	$A \vee B$	$A \wedge (A \vee B)$	φ_1
T	T	T	T	T
T	F	T	T	T
F	T	T	F	T
F	F	F	F	T

   (b) $\varphi_2 : ((A \vee C) \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee C)$

A	B	C	$A \vee C$	$B \vee C$	$(A \vee C) \wedge (B \vee C)$	$A \wedge B$	$(A \wedge B) \vee C$	φ_2
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	T	T	T	F	T	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

   (c) $\varphi_3 : (A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

A	B	C	$A \Rightarrow B$	$B \Rightarrow C$	$A \Rightarrow C$	$(B \Rightarrow C) \Rightarrow (A \Rightarrow C)$	φ_3
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

2. Give Natural Deduction proof trees for each of the above propositions.

$$\dfrac{\dfrac{\dfrac{}{\{A\}\vdash A}\text{Hyp} \quad \dfrac{\dfrac{}{\{A\}\vdash A}\text{Hyp}}{\{A\}\vdash A\lor B}\text{Or}_L\text{ I}}{\{A\}\vdash A\land(A\lor B)}\text{And I}}{\{\}\vdash A\Rightarrow A\land(A\lor B)}\text{Imp I}$$

Let $\Gamma_1 = \{(A\lor C)\land(B\lor C), A\lor C\}$ and $\Gamma_2 = \Gamma_1\cup\{A, B\lor C\}$ in

$$\dfrac{\dfrac{\dfrac{}{\{(A\lor C)\land(B\lor C)\}\vdash (A\lor C)\land(B\lor C)}\text{Hyp} \quad \dfrac{\dfrac{}{\Gamma_1\vdash A\lor C}\text{Hyp} \quad P_1 \quad \dfrac{\dfrac{}{\Gamma_1\cup\{C\}\vdash C}\text{Hyp}}{\Gamma_1\cup\{C\}\vdash (A\land B)\lor C}\text{Or}_R\text{ I}}{\Gamma_1\vdash(A\land B)\lor C}\text{Or E}}{\{(A\lor C)\land(B\lor C)\}\vdash (A\land B)\lor C}\text{And}_L\text{ E}}{\{\}\vdash ((A\lor C)\land(B\lor C))\Rightarrow((A\land B)\lor C)}\text{Imp I}$$

where $P_1 =$

$$\dfrac{\dfrac{}{\Gamma_1\cup\{A\}\vdash (A\lor C)\land(B\lor C)}\text{Hyp} \quad \dfrac{\dfrac{}{\Gamma_2\vdash B\lor C}\text{Hyp} \quad \dfrac{\dfrac{\dfrac{}{\Gamma_2\cup\{B\}\vdash A}\text{Hyp} \quad \dfrac{}{\Gamma_2\cup\{B\}\vdash B}\text{Hyp}}{\Gamma_2\cup\{B\}\vdash (A\land B)}\text{And I}}{\Gamma_2\cup\{B\}\vdash (A\land B)\lor C}\text{Or}_L\text{ I} \quad \dfrac{\dfrac{}{\Gamma_2\cup\{C\}\vdash C}\text{Hyp}}{\Gamma_2\cup\{C\}\vdash (A\land B)\lor C}\text{Or}_R\text{ I}}{\Gamma_2\vdash (A\land B)\lor C}\text{Or E}}{\Gamma_1\cup\{A\}\vdash (A\land B)\lor C}\text{And}_R\text{ E}$$

Finally, let $\Gamma_1 = \{A\Rightarrow B, B\Rightarrow C\}$ in

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1\cup\{A\}\vdash A\Rightarrow B}\text{Hyp} \quad \dfrac{}{\Gamma_1\cup\{A\}\vdash A}\text{Hyp} \quad \dfrac{\dfrac{}{\Gamma_1\cup\{A,B\}\vdash B\Rightarrow C}\text{Hyp} \quad \dfrac{}{\Gamma_1\cup\{A,B\}\vdash B}\text{Hyp} \quad \dfrac{}{\Gamma_1\cup\{A,B,C\}\vdash C}\text{Hyp}}{\Gamma_1\cup\{A,B\}\vdash C}\text{Imp E}}{\Gamma_1\cup\{A\}\vdash C}\text{Imp E}}{\Gamma_1\vdash A\Rightarrow C}\text{Imp I}}{\{A\Rightarrow B\}\vdash (B\Rightarrow C)\Rightarrow(A\Rightarrow C)}\text{Imp I}}{\{\}\vdash (A\Rightarrow B)\Rightarrow((B\Rightarrow C)\Rightarrow(A\Rightarrow C))}\text{Imp I}$$

3. Consider the signature $\mathcal{G} = (V = \{x, y, z\}, F = \emptyset, af = \emptyset, R = \{<\}, ar = \{< \mapsto 2\})$, and the structure $S = \{\mathcal{G}, \mathcal{D} = \mathbb{N}, \mathcal{F}, \phi, \mathcal{R}, \rho\}$ where $\rho(<)$ is normal less-than comparison. Given the formula $\forall y.\ y < x \Rightarrow (\forall z.\ \neg(z < y))$:

   (a) give an assignment that satisfies the formula, and explain why it does.
   Any assignment that assigns 1 to x will do, since the only $y < 1$ is 0, and there is no $z < 0$ in $\mathbb{N}$. Any assignment that assigns 0 to x will also work, since there is no $y < 0$, making the implication vacuously true.

   (b) give an assignment that falsifies the formula, and explain why it does.
   Any assignment that assigns a number greater than 1 to x will do, since we can then choose $y < x$ such that $y > 0$, and then choose 0 for z to falsify the formula.

4. Give a Natural Deduction proof for the formula $(\exists x.\ \forall y.\ r(x,y)) \Rightarrow (\forall y.\ \exists x.\ r(x,y))$

$$\dfrac{\dfrac{}{\{\exists x.\ \forall y.\ r(x,y)\} \vdash \exists x.\ \forall y.\ r(x,y)}\textsc{Hyp} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\{\exists x.\ \forall y.\ r(x,y), \forall y.\ r(x,y)\} \vdash \forall y.\ r(x,y)}\textsc{Hyp} \qquad \dfrac{}{\{\exists x.\ \forall y.\ r(x,y), \forall y.\ r(x,y), r(x,y)\} \vdash r(x,y)}\textsc{Hyp}}{\{\exists x.\ \forall y.\ r(x,y), \forall y.\ r(x,y)\} \vdash r(x,y)}\textsc{All E}}{\{\exists x.\ \forall y.\ r(x,y), \forall y.\ r(x,y)\} \vdash \exists x.\ r(x,y)}\textsc{Ex I}}{\{\exists x.\ \forall y.\ r(x,y), \forall y.\ r(x,y)\} \vdash \forall y.\ \exists x.\ r(x,y)}\textsc{All I}}{\dfrac{\{\exists x.\ \forall y.\ r(x,y)\} \vdash \forall y.\ \exists x.\ r(x,y)}{\{\} \vdash (\exists x.\ \forall y.\ r(x,y)) \Rightarrow (\forall y.\ \exists x.\ r(x,y))}\textsc{Imp I}}\textsc{Ex E}$$

5. Prove the following Hoare triple:

$$\{n \geq 1\} f := 1;\ i := n;\ \texttt{while}\ i > 1\ \texttt{do}\ (f := f * i;\ i := i - 1)\{f = n!\}$$

By Precondition Strengthening, it suffices to show that

$$\{n \geq 1 \wedge 1 = n!/n!\} f := 1;\ i := n;\ \texttt{while}\ i > 1\ \texttt{do}\ (f := f * i;\ i := i - 1)\{f = n!\}$$

The reason for this rather strange precondition will become clear when we reach the `while` loop. Using the Sequencing rule and the Assignment Rule, we can show that

$$\{n \geq 1 \wedge 1 = n!/n!\} f := 1 \{n \geq 1 \wedge f = n!/n!\}$$

and

$$\{n \geq 1 \wedge f = n!/n!\} i := n \{i \geq 1 \wedge f = n!/i!\}$$

leaving us with the remaining triple

$$\{i \geq 1 \wedge f = n!/i!\} \texttt{while}\ i > 1\ \texttt{do}\ (f := f * i;\ i := i - 1)\{f = n!\}$$

By Postcondition Weakening, it is sufficient to show that

$$\{i \geq 1 \wedge f = n!/i!\} \texttt{while}\ i > 1\ \texttt{do}\ (f := f * i;\ i := i - 1)\{i \geq 1 \wedge f = n!/i! \wedge i \leq 1\}$$

since $i \geq 1 \wedge i \leq 1$ implies that $i = 1$, and $n!/1! = n!$. Now we can use the While rule: by showing that

$$\{i \geq 1 \wedge f = n!/i! \wedge i > 1\} f := f * i;\ i := i - 1 \{i \geq 1 \wedge f = n!/i!\}$$

we will have verified the original triple. $i > 1$ implies that $i - 1 \geq 1$, and $f = n!/i!$ implies that $f * i = n!/(i-1)!$, so by Precondition Strengthening, this is equivalent to showing that

$$\{i - 1 \geq 1 \wedge f * i = n!/(i-1)!\} f := f * i;\ i := i - 1 \{i \geq 1 \wedge f = n!/i!\}$$

Now, by the Assignment rule we show that

$$\{i - 1 \geq 1 \wedge f * i = n!/(i-1)!\} f := f * i \{i - 1 \geq 1 \wedge f = n!/(i-1)!\}$$

and

$$\{i - 1 \geq 1 \wedge f = n!/(i-1)!\} i := i - 1 \{i \geq 1 \wedge f = n!/i!\}$$

and use the Sequencing Rules to combine these two steps, completing the proof.

6. Give a Floyd-Hoare rule for the command `repeat` C `until` B, which repeatedly executes C until B is true. Note that B is checked *after* each execution of C, so that C is always executed at least once.

   The simplest correct rule is:

$$\frac{\{P\}C\{Q\} \qquad \{Q \wedge \neg B\}C\{Q\}}{\{P\}\texttt{repeat}\ C\ \texttt{until}\ B\{Q \wedge B\}}$$

   More complicated solutions are possible; the rules are correct as long as they reflect the fact that `repeat` C `until` $B = C$; `while` $\neg B$ `do` C.

7. Calculate weakest preconditions and verification conditions for the following Hoare triples:

   (a) $\{n \geq 1\}f := 1;\ i := n;\ \texttt{while}\ i > 1\ \texttt{do}\ (f := f * i;\ i := i - 1)\{f = n!\}$

   First, we have to annotate the loop with its loop invariant, $P = (i \geq 1 \wedge f = n!/i!)$. Now: $wp\ (\texttt{while}\ i > 1\ \texttt{inv}\ P\ \texttt{do}\ (f := f * i;\ i := i - 1))\ \{f = n!\} = P$, $wp\ (i := n;\ \texttt{while}\ i > 1\ \texttt{inv}\ P\ \texttt{do}\ (f := f * i;\ i := i - 1))\ \{f = n!\} = wp\ (i := n)\ P = P[i \Rightarrow n]$, and the weakest precondition for the entire program is $P[i \Rightarrow n, f \Rightarrow 1] = (n \geq 1 \wedge 1 = n!/n!)$ (which you might recognize from the proof above).

   For verification conditions, since the conditions generated by a sequence of assignment statements is `true`, the conditions for the entire program are equal to $vcg\ (\texttt{while}\ i > 1\ \texttt{inv}\ P\ \texttt{do}\ (f := f * i;\ i := i - 1))\ \{f = n!\} = ((P \wedge i > 1) \Rightarrow wp\ C\ P) \wedge (vcg\ C\ P) \wedge ((P \wedge i \leq 1) \Rightarrow (f = n!))$, where C is the loop body. $wp\ C\ P$ is $P[f \Rightarrow f * i][i \Rightarrow i - 1]$, and $vcg\ C\ P$ is `true`, so the above is equivalent to $((P \wedge i > 1) \Rightarrow P[f \Rightarrow f * i][i \Rightarrow i - 1]) \wedge ((P \wedge i \leq 1) \Rightarrow (f = n!))$.

   (b) $\{a > 0 \wedge b > 0\}$
   $m := a;\ n := b;$
   $\texttt{while}\ n \neq m\ \texttt{do}\ (\texttt{if}\ m < n\ \texttt{then}\ n := n - m\ \texttt{else}\ m := m - n)$
   $\{a \bmod m = 0 \wedge b \bmod m = 0\}$

   A likely loop invariant for this program's loop is $P = (\text{GCD}(a, b) = \text{GCD}(n, m))$. With this invariant, the weakest precondition for the program is $P[n \Rightarrow b][m \Rightarrow a] = (\text{GCD}(a, b) = \text{GCD}(a, b))$, which is trivial as long as $a > 0$ and $b > 0$.

   As before, the assignment statements contribute no verification conditions, and the vcg of the program is $((P \wedge m \neq n) \Rightarrow (wp\ C\ P)) \wedge ((P \wedge m = n) \Rightarrow (a \bmod m = 0 \wedge b \bmod m = 0))$, where C is the loop body. Now $wp\ C\ P = (m < n \wedge (P[n \Rightarrow n - m])) \vee (m \geq n \wedge (P[m \Rightarrow m - n]))$, giving a final verification condition of $((P \wedge m \neq n) \Rightarrow ((m < n \wedge (P[n \Rightarrow n - m])) \vee (m \geq n \wedge (P[m \Rightarrow m - n])))) \wedge ((P \wedge m = n) \Rightarrow (a \bmod m = 0 \wedge b \bmod m = 0))$.